

DOMINANT WEIGHT MULTIPLICITIES IN HYBRID CHARACTERS OF B_n, C_n, F_4, G_2

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ABSTRACT. The characters of irreducible finite dimensional representations of compact simple Lie group G are invariant with respect to the action of the Weyl group $W(G)$ of G . The defining property of the new character-like functions ('hybrid characters') is the fact that $W(G)$ acts differently on the character term corresponding to the long roots than on those corresponding to the short roots. Therefore the hybrid characters are defined for the simple Lie groups with two different lengths of their roots. Dominant weight multiplicities for the hybrid characters are determined. The formulas for 'hybrid dimensions' are also found for all cases as the zero degree term in power expansion of the 'hybrid characters'.

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1. INTRODUCTION

Motivation to study properties of characters of irreducible representations of simple Lie groups was justified by their fundamental importance for the representation theory [1, 16]. Subsequently one may find the motivation in other properties of the characters, such as the possibility to discretize them uniformly for simple Lie groups of all types and ranks [13, 14] and, more generally, the duality between the representation theory and the conjugacy classes of elements of finite order [13, 14] in compact simple Lie groups.

Consequently one can discretize a vast class of orthogonal special functions and polynomials related to characters. With a twist of familiar definition of degree of multivariable polynomial, their nodes and the cubature formulas appear naturally and are optimal (called Gaussian) in their efficiency [12, 17, 18].

Corresponding Fourier transforms of digital data are yet to be exploited. The hybrid characters of this paper lead to new classes of special functions that are orthogonal on lattices of all dimensions and symmetry types. For that to be used, one needs the dominant weight multiplicities of the hybrid characters for every representation that comes into considerations.

The later motivation gains in importance with ever increasing number of digital data that require processing and that are collected today on lattices of various densities and all possible symmetries in dimensions ≥ 2 .

In general, the character of a finite dimensional representation of a compact semisimple Lie group (8) is a finite sum of exponential functions with exponents depending through the scalar product $\langle \lambda, x \rangle$ on the points x of the maximal torus of the Lie group and on a weight λ of the weight system of the representation. For many years a practical obstacle in exploiting the characters of other than a few

lowest representations, was the relative inefficiency of the general algorithm [3] for computing the multiplicities of weights λ of the representation. This obstacle was removed by the fast algorithm [15, 2]. Truly large scale applications involving the characters became possible (for example [4]).

In this paper the problem of multiplicities of weights is solved for the recently discovered character-like functions for any simple Lie group with roots of two different lengths. More precisely, the present paper solves the problem of determining the coefficients $p_\mu^\lambda(\mathfrak{g})$ and $q_\mu^\lambda(\mathfrak{g})$ arising in (11) and (12), when the hybrid characters are written as linear combinations of the C -functions. Once those coefficients are known, the discretization of the C -functions [13, 14] can be directly extended to the hybrid characters.

The new classes of character-like functions are formed by defining two new homomorphisms from the Weyl group $W(\mathfrak{g})$ of the simple Lie algebra \mathfrak{g} to $\{\pm 1\}$ which generalize the usual sign function.

$$\sigma^L, \sigma^S : W(\mathfrak{g}) \longrightarrow \{\pm 1\}, \quad (1)$$

In this paper we consider the cases of Lie groups/Lie algebras of types B_n , C_n of any rank n , as well as F_4 and G_2 . The homomorphisms (1) are defined by distinguishing their values on the reflections with respect to hyperplanes orthogonal to long and short roots of \mathfrak{g} . Such a possibility was first noted in [11] in the case of C_2 , while considering the 2-variable specialization of the n -variable case [8]. In [12] the hybrid characters are described for all simple Lie algebras with 2-root lengths. The definitions use the homomorphisms (1) denoted by σ^L and σ^S . Also the G_2 case is found in [17]. For every representation of those groups there are two new character-like functions, see (11) and (12), in addition to their corresponding character (8) functions.

The role of irreducible characters in representation theory of simple Lie groups cannot be overestimated. Therefore it is interesting to investigate the structure and the role of the new ‘hybrid’ characters that so closely resemble the well known irreducible character functions. Regardless of whether one is able to link them to some algebraic structures, their orthogonality opens new possibilities of applications in Fourier analysis.

Among the properties of the character (8) that are also shared with the hybrid characters (11) and (12), let us point out the following ones:

- Completeness as the bases in their respective functional spaces [12].
- Orthogonality [5] when integrated over the fundamental region $F(\mathfrak{g})$.
- Decomposition of products of the functions into their sums [11].
- Discrete orthogonality [5] when summed up over a fragment of a lattice $F_M(\mathfrak{g})$ in $F(\mathfrak{g})$, where density of the lattice is fixed by our choice of $M \in \mathbb{N}$.

Properties of the corresponding Fourier transforms are being investigated, see [5, 11, 12] and references therein.

Another useful application of the hybrid characters is a consequence of their well-defined behaviour at the boundary ∂F of their domain F . While irreducible characters and the C -functions are symmetric at ∂F , the S -functions are skew symmetric at ∂F . The hybrid characters are symmetric and skew symmetric at different parts of ∂F . Hence Fourier expansions of functions with similar boundary behaviour would be most simply done in terms of expansions into series of corresponding hybrid characters.

2. PRELIMINARIES

Let \mathfrak{g} be one of the simple Lie algebras

$$B_n \ (n \geq 2), \quad C_n \ (n \geq 2), \quad F_4, \quad G_2. \quad (2)$$

The Lie algebras B_2 and C_2 are isomorphic but it is convenient to keep both of them. We adopt the Dynkin numbering of simple roots, see for example [2].

In this section we establish a convenient notation for the root systems and Weyl groups for each of these algebras that we will use in subsequent sections. Let $\{e_1, \dots, e_n\}$ denote the orthonormal basis (e -basis) of the real Euclidean space \mathbb{R}^n of dimension n . We express in the e -basis expansions for a base of simple roots (α -basis), and the corresponding fundamental weights (ω -basis) of \mathfrak{g} . In particular, we know the Cartan matrix \mathfrak{C} for any \mathfrak{g} .

$$\mathfrak{C} = \left(2 \frac{\langle \alpha_j | \alpha_k \rangle}{\langle \alpha_k | \alpha_k \rangle} \right), \quad j, k = 1, 2, \dots, n.$$

where $\langle \cdot | \cdot \rangle$ is the usual scalar product in \mathbb{R}^n . These are well known [1, 2] standard attributes of any simple \mathfrak{g} . The matrix \mathfrak{C} links the α - and ω -bases,

$$\alpha = \mathfrak{C}\omega, \quad \omega = \mathfrak{C}^{-1}\alpha.$$

A reflection r_ξ with respect to a hyperplane orthogonal to ξ and containing the origin in \mathbb{R}^n , is given by

$$r_\xi x = x - \frac{2\langle x | \xi \rangle}{\langle \xi | \xi \rangle} \xi, \quad x, \xi \in \mathbb{R}^n. \quad (3)$$

The Weyl group of \mathfrak{g} , denoted $W(\mathfrak{g})$, is the reflection group generated by $\{r_{\alpha_i}\}$.

2.1. The Lie algebra B_n .

The e -basis realization of a base of simple roots and the corresponding fundamental weights of B_n is given by

simple roots	fundamental weights
$\alpha_1 = e_1 - e_2$	$\omega_1 = e_1$
$\alpha_2 = e_2 - e_3$	$\omega_2 = e_1 + e_2$
\vdots	\vdots
$\alpha_{n-1} = e_{n-1} - e_n$	$\omega_{n-1} = e_1 + \dots + e_{n-1}$
$\alpha_n = e_n$	$\omega_n = \frac{1}{2}(e_1 + \dots + e_n)$

Let ρ^L (respectively ρ^S) denote half the sum of the long (respectively short) positive roots of B_n . Then we have

$$\rho^L = \omega_1 + \dots + \omega_{n-1}, \quad \rho^S = \omega_n. \quad (4)$$

Recall that the Weyl group $W(B_n)$ of B_n can be viewed as the set of all permutations and sign changes on the subscripts $\{1, \dots, n\}$ of the e -basis. With this identification, if σ^L and σ^S denote the long and short sign homomorphisms from $W(B_n)$ to $\{\pm 1\}$, we have:

$r_{\alpha_1} = (1, 2)$	$\sigma^L(r_{\alpha_1}) = -1$	$\sigma^S(r_{\alpha_1}) = +1$
$r_{\alpha_2} = (2, 3)$	$\sigma^L(r_{\alpha_2}) = -1$	$\sigma^S(r_{\alpha_2}) = +1$
\vdots	\vdots	\vdots
$r_{\alpha_{n-1}} = (n-1, n)$	$\sigma^L(r_{\alpha_{n-1}}) = -1$	$\sigma^S(r_{\alpha_{n-1}}) = +1$
$r_{\alpha_n} = n \rightarrow -n$	$\sigma^L(r_{\alpha_n}) = +1$	$\sigma^S(r_{\alpha_n}) = -1$

The set of long roots in the root system of B_n is a root system equivalent to the root system of D_n . The set of short roots of the root system of B_n is a root system equivalent to the root system of $nA_1 := A_1 \oplus \dots \oplus A_1$.

2.2. The Lie algebra C_n .

The e -basis realization of a base of simple roots and the corresponding fundamental weights of C_n is given by

simple roots	fundamental weights
$\alpha_1 = e_1 - e_2$	$\omega_1 = e_1$
$\alpha_2 = e_2 - e_3$	$\omega_2 = e_1 + e_2$
\vdots	\vdots
$\alpha_{n-1} = e_{n-1} - e_n$	$\omega_{n-1} = e_1 + \cdots + e_{n-1}$
$\alpha_n = 2e_n$	$\omega_n = e_1 + \cdots + e_n$

Let ρ^L (respectively ρ^S) denote half sum of the long (respectively short) positive roots of C_n . Then we have

$$\rho^L = \omega_n, \quad \rho^S = \omega_1 + \cdots + \omega_{n-1}. \quad (5)$$

Recall that the Weyl group $W(C_n)$ of C_n can be viewed as the set of all permutations and sign changes on the subscripts $\{1, \dots, n\}$ of the e -basis. With this identification, if σ^L and σ^S denote the long and short sign homomorphisms from $W(C_n)$ to $\{\pm 1\}$, we have:

$r_{\alpha_1} = (1, 2)$	$\sigma^L(r_{\alpha_1}) = +1$	$\sigma^S(r_{\alpha_1}) = -1$
$r_{\alpha_2} = (2, 3)$	$\sigma^L(r_{\alpha_2}) = +1$	$\sigma^S(r_{\alpha_2}) = -1$
\vdots	\vdots	\vdots
$r_{\alpha_{n-1}} = (n-1, n)$	$\sigma^L(r_{\alpha_{n-1}}) = +1$	$\sigma^S(r_{\alpha_{n-1}}) = -1$
$r_{\alpha_n} = n \rightarrow -n$	$\sigma^L(r_{\alpha_n}) = -1$	$\sigma^S(r_{\alpha_n}) = +1$

The set of long roots in the root system of C_n is a root system equivalent to the root system of nA_n . The set of short roots of the root system of C_n is a root system equivalent to the root system of D_n .

2.3. The Lie algebra F_4 .

The e -basis realization of a base of simple roots and the corresponding fundamental weights of F_4 is given by

simple roots	fundamental weights
$\alpha = e_2 - e_3$	$\omega_1 = e_1 + e_2$
$\beta = e_3 - e_4$	$\omega_2 = 2e_1 + e_2 + e_3$
$\gamma = e_4$	$\omega_3 = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4)$
$\delta = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$	$\omega_4 = e_1$

Let ρ^L (respectively ρ^S) denote one half the sum of the long (respectively short) positive roots of F_4 , then we have

$$\rho^L = \omega_1 + \omega_2 \quad \text{and} \quad \rho^S = \omega_3 + \omega_4.$$

The Weyl group $W(F_4)$ of F_4 can be viewed as the set of all permutations and sign changes on the subscripts $\{1, 2, 3, 4\}$ together with the maps sending the basis $\{e_1, \dots, e_4\}$ to an orthogonal basis of vectors in $\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$.

If σ^L and σ^S denote the long and short sign homomorphisms from $W(F_4)$ to $\{\pm 1\}$ we have:

$$\begin{array}{lll}
r_\alpha = (2, 3) & & \\
r_\beta = (3, 4) & & \\
r_\gamma = 4 \rightarrow -4 & \sigma^L(r_\alpha) = -1 & \sigma^S(r_\alpha) = +1 \\
r_\delta = \begin{cases} e_1 \rightarrow \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \\ e_2 \rightarrow \frac{1}{2}(e_1 + e_2 - e_3 - e_4) \\ e_3 \rightarrow \frac{1}{2}(e_1 - e_2 + e_3 - e_4) \\ e_4 \rightarrow \frac{1}{2}(e_1 - e_2 - e_3 + e_4) \end{cases} & \begin{array}{l} \sigma^L(r_\beta) = -1 \\ \sigma^L(r_\gamma) = +1 \\ \sigma^L(r_\delta) = +1 \end{array} & \begin{array}{l} \sigma^S(r_\beta) = +1 \\ \sigma^S(r_\gamma) = -1 \\ \sigma^S(r_\delta) = -1 \end{array}
\end{array}$$

The set of long roots in the root system of F_4 is a root system equivalent to the root system of D_4 . The set of short roots of the root system of F_4 is a root system equivalent to the root system of D_4 .

2.4. The Lie algebra G_2 .

The e -basis realization of a base of simple roots and the corresponding fundamental weights of G_2 is given by

$$\begin{array}{ll}
\text{simple roots} & \text{fundamental weights} \\
\alpha = e_1 - e_2 & \omega_1 = e_1 - e_3 \\
\beta = \frac{1}{3}(-e_1 + 2e_2 - e_3) & \omega_2 = \frac{1}{3}(e_1 + e_2 - 2e_3)
\end{array}$$

Let ρ^L (respectively ρ^S) denote one half the sum of the long (respectively short) positive roots of G_2 , then we have

$$\rho^L = \omega_1 \text{ and } \rho^S = \omega_2.$$

Recall that the Weyl group $W(G_2)$ of G_2 is generated by the set of all permutations and minus the identity on the subscripts $\{1, 2, 3\}$. With this identification if σ denotes the usual sign function on $W(G_2)$ and σ^L, σ^S denote the long and short sign homomorphisms from $W(G_2)$ to $\{\pm 1\}$ we have:

$$\begin{array}{llll}
r_\alpha = (1, 2) & \sigma(r_\alpha) = -1 & \sigma^L(r_\alpha) = +1 & \sigma^S(r_\alpha) = -1 \\
r_\beta = (-id) \circ (1, 3) & \sigma(r_\beta) = -1 & \sigma^L(r_\beta) = -1 & \sigma^S(r_\beta) = +1
\end{array}$$

The set of long roots in the root system of G_2 is a root system equivalent to the root system of A_2 . The set of short roots of the root system of G_2 is a root system equivalent to the root system of A_2 .

3. HYBRID CHARACTER FUNCTIONS

Let $P^+(\mathfrak{g})$ denote the dominant integral weights of \mathfrak{g} , then for any $\lambda \in P^+(\mathfrak{g})$, the W -invariant C -functions and the W -skew invariant S -functions are defined as follows,

$$C_\lambda(\mathfrak{g})(x) = \sum_{\mu \in W(\mathfrak{g}) \cdot \lambda} e^{2\pi i \langle \mu | x \rangle} \quad (6)$$

$$S_{\rho+\lambda}(\mathfrak{g})(x) = \sum_{\mu \in W(\mathfrak{g}) \cdot (\lambda + \rho)} \sigma(\mu) e^{2\pi i \langle \mu | x \rangle} \quad (7)$$

where $\sigma(\mu) = \sigma(\phi)$ when $\mu = \phi \cdot (\lambda + \rho)$. The irreducible character function is then given by

$$\chi_\lambda(\mathfrak{g})(x) = \frac{S_{\lambda+\rho}(\mathfrak{g})(x)}{S_\rho(\mathfrak{g})(x)}$$

and the Weyl character formula expresses the character function as a linear combination of C -functions

$$\chi_\lambda(\mathfrak{g})(x) = \frac{S_{\rho+\lambda}(\mathfrak{g})(x)}{S_\rho(\mathfrak{g})(x)} = \sum_{\mu} m_{\mu}^{\lambda}(\mathfrak{g}) C_{\mu}(\mathfrak{g})(x) \quad (8)$$

where, for each dominant integral weight μ of \mathfrak{g} , $m_{\mu}^{\lambda}(\mathfrak{g})$ is the dominant weight multiplicity. It denotes the dimension of the μ weight space in the simple \mathfrak{g} module having highest weight λ .

The hybrid S -functions are defined as follows,

$$S_{\rho^L+\lambda}^L(\mathfrak{g})(x) = \sum_{\nu \in W(\mathfrak{g}) \cdot (\rho^L+\lambda)} \sigma^L(\nu) e^{2\pi i \langle \nu | x \rangle} \quad (9)$$

$$S_{\rho^S+\lambda}^S(\mathfrak{g})(x) = \sum_{\nu \in W(\mathfrak{g}) \cdot (\rho^S+\lambda)} \sigma^S(\nu) e^{2\pi i \langle \nu | x \rangle} \quad (10)$$

with $\sigma^L(\nu) = \sigma^L(\phi)$ (respectively $\sigma^S(\nu) = \sigma^S(\phi)$) where $\nu = \phi \cdot (\rho^L + \lambda)$ (respectively $\nu = \phi \cdot (\rho^S + \lambda)$).

The corresponding hybrid character functions are given by

$$\chi_{\lambda}^L(\mathfrak{g})(x) = \frac{S_{\rho^L+\lambda}^L(\mathfrak{g})(x)}{S_{\rho^L}^L(\mathfrak{g})(x)} = \sum_{\mu} p_{\mu}^{\lambda}(\mathfrak{g}) C_{\mu}(\mathfrak{g})(x) \quad (11)$$

$$\chi_{\lambda}^S(\mathfrak{g})(x) = \frac{S_{\rho^S+\lambda}^S(\mathfrak{g})(x)}{S_{\rho^S}^S(\mathfrak{g})(x)} = \sum_{\mu} q_{\mu}^{\lambda}(\mathfrak{g}) C_{\mu}(\mathfrak{g})(x) \quad (12)$$

where the summations range over the dominant integral weights μ of \mathfrak{g} .

While the coefficients $m_{\mu}^{\lambda}(\mathfrak{g})$ in the expansion of the irreducible characters, $\chi_{\lambda}(\mathfrak{g})(x)$, are calculated by the known algorithm [15] and are tabulated in [2], the coefficients $p_{\mu}^{\lambda}(\mathfrak{g})$ and $q_{\mu}^{\lambda}(\mathfrak{g})$ are studied here for the first time.

4. LONG ROOT HYBRID CHARACTER FUNCTIONS

In this section we express the coefficients $p_{\mu}^{\lambda}(\mathfrak{g})$ of (11) in terms of the multiplicities m_{μ}^{λ} of (8). Since m_{μ}^{λ} can be efficiently calculated [15, 2], we have effectively determined the values of $p_{\mu}^{\lambda}(\mathfrak{g})$.

Let \mathfrak{g} denote one of the simple Lie algebras B_n, C_n, F_4, G_2 . Let \mathcal{G}_S denote the subgroup of the Weyl group $W(\mathfrak{g})$ generated by the reflections r_{α} where α ranges over the short simple roots of \mathfrak{g} . We now list a number of properties of this subgroup.

1: By the definition of the Weyl group $W(\mathfrak{g})$, we have that \mathcal{G}_S is a transversal of the Weyl group $W(\Phi_L)$ of the long root subsystem Φ_L in the Weyl group $W(\mathfrak{g})$ —i.e. $W(\mathfrak{g}) = \bigcup_{\phi \in \mathcal{G}_S} W(\Phi_L) \cdot \phi$.

2: By the definition of the sign homomorphism σ^L , we have that $\sigma^L(\phi) = +1$ for all $\phi \in \mathcal{G}_S$.

3: There exists a base of simple roots Δ_L in Φ_L such that each $\phi \in \mathcal{G}_S$ permutes these simple roots. In fact we have

\mathfrak{g}	Φ_L	\mathcal{G}_S	Δ_L
B_n	D_n	$\langle r_n \rangle$	$\{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$
C_n	nA_1	$\langle r_{\alpha_1}, \dots, r_{\alpha_{n-1}} \rangle$	$\{2e_1, \dots, 2e_n\}$
F_4	D_4	$\langle r_{\gamma}, r_{\delta} \rangle$	$\{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4\}$
G_2	A_2	$\langle r_{\beta} \rangle$	$\{e_1 - e_2, e_2 - e_3\}$

For each case let $\{\omega'_i\}$ denote the fundamental weights corresponding to the base of simple roots Δ_L . In the table below we provide the expansions for dominant

integral weights λ of \mathfrak{g} in terms of the $\{\omega'_i\}$ basis. In particular we observe that each dominant integral weight of \mathfrak{g} is a dominant integral weight of Φ_L . In fact we have

\mathfrak{g}	λ	Φ_L	λ
B_n	$\sum_{i=1}^n N_i \omega_i$	D_n	$\sum_{i=1}^{n-1} N_i \omega'_i + (N_{n-1} + N_n) \omega'_n$
C_n	$\sum_{i=1}^n N_i \omega_i$	nA_1	$\sum_{i=1}^n \sum_{j=i}^n N_j \omega'_i$
F_4	$\sum_{i=1}^4 N_i \omega_i$	D_4	$(N_2 + N_3 + N_4) \omega'_1 + N_1 \omega'_2 + N_2 \omega'_3 + (N_2 + N_3) \omega'_4$
G_2	$N_1 \omega_1 + N_2 \omega_2$	A_2	$N_1 \omega'_1 + (N_1 + N_2) \omega'_2$

TABLE 1. Dominant integral weight λ in ω -basis for the Lie algebras \mathfrak{g} is written in ω'_i -basis for Φ_L .

4: Since each $\phi \in \mathcal{G}_S$ permutes the base of simple roots Δ_L , ϕ also permutes the associated fundamental weights $\{\omega'_i\}$ of Φ_L and hence if λ is a dominant integral weight of \mathfrak{g} then $\phi \cdot \lambda$ is also a dominant integral weight of Φ_L . In particular we have that $\phi \cdot \rho^L = \rho^L$.

Using the properties of \mathcal{G}_S listed above we have that for any dominant weight λ of \mathfrak{g}

$$C_\lambda^L(\mathfrak{g})(x) = \sum_{\mu \in \mathcal{G}_S \cdot \lambda} C_\mu(\Phi_L)(x)$$

$$S_{\rho^L + \lambda}^L(\mathfrak{g})(x) = \sum_{\mu \in \mathcal{G}_S \cdot \lambda} S_{\rho^L + \mu}(\Phi_L)(x)$$

and finally

$$\chi_\lambda^L(\mathfrak{g})(x) = \frac{S_{\rho^L + \lambda}^L(\mathfrak{g})(x)}{S_{\rho^L}^L(\mathfrak{g})(x)} = \sum_{\mu \in \mathcal{G}_S \cdot \lambda} \chi_\mu(\Phi_L)(x)$$

Since $\chi_\mu(\Phi_L)(x) = \sum_{\nu} m_\nu^\mu(\Phi_L) C_\nu(\Phi_L)(x)$ where $m_\nu^\mu(\Phi_L)$ denotes the multiplicity of the dominant integral weight ν in the simple module of the Lie algebra associated with Φ_L having highest weight μ . Using this we conclude that

$$p_\nu^\lambda(\mathfrak{g}) = \sum_{\mu \in \mathcal{G}_S \cdot \lambda} m_\nu^\mu(\Phi_L). \quad (13)$$

Example 1. Consider the hybrid character function $\chi_\lambda^L(G_2)(x)$ where $\lambda = 2\omega_1 + 2\omega_2$. If ω'_i denote the fundamental weights with respect to Δ_L we have that $N_1\omega + N_2\omega_2 = N_1\omega'_1 + (N_1 + N_2)\omega'_2$ and $r_\beta \cdot (N_1\omega_1 + N_2\omega_2) = (N_1 + N_2)\omega'_1 + N_1\omega'_2$. For notational convenience we represent the dominant integral weights using coordinates with respect to the fundamental weights of the appropriate algebras. Applying the formula given in (13) we obtain

$$\begin{aligned}
p_{(2,2)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(2,4)}^\nu(A_2) = m_{(2,4)}^{(2,4)}(A_2) + m_{(2,4)}^{(4,2)}(A_2) = 1 + 0 = 1 \\
p_{(0,5)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(0,5)}^\nu(A_2) = m_{(0,5)}^{(2,4)}(A_2) + m_{(0,5)}^{(4,2)}(A_2) = 1 + 0 = 1 \\
p_{(2,1)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(2,3)}^\nu(A_2) = m_{(2,3)}^{(2,4)}(A_2) + m_{(2,3)}^{(4,2)}(A_2) = 0 + 1 = 1 \\
p_{(0,4)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(0,4)}^\nu(A_2) = m_{(0,4)}^{(2,4)}(A_2) + m_{(0,4)}^{(4,2)}(A_2) = 0 + 1 = 1 \\
p_{(1,2)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(1,3)}^\nu(A_2) = m_{(1,3)}^{(2,4)}(A_2) + m_{(1,3)}^{(4,2)}(A_2) = 2 + 0 = 2 \\
p_{(1,1)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(1,2)}^\nu(A_2) = m_{(1,2)}^{(2,4)}(A_2) + m_{(1,2)}^{(4,2)}(A_2) = 0 + 2 = 2 \\
p_{(0,2)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(0,2)}^\nu(A_2) = m_{(0,2)}^{(2,4)}(A_2) + m_{(0,2)}^{(4,2)}(A_2) = 3 + 0 = 3 \\
p_{(0,1)}^{(2,2)}(G_2) &= \sum_{(2,4) \in \{(2,4), (4,2)\}} m_{(0,1)}^\nu(A_2) = m_{(0,1)}^{(2,4)}(A_2) + m_{(0,1)}^{(4,2)}(A_2) = 0 + 3 = 3
\end{aligned}$$

These calculations imply that

$$\begin{aligned}
\chi_{(2,2)}^L(x) &= C_{(2,2)}(x) + C_{(0,5)}(x) + C_{(2,1)}(x) + C_{(0,4)}(x) + 2C_{(1,2)}(x) \\
&\quad + 2C_{(1,1)}(x) + 3C_{(0,2)}(x) + 3C_{(0,1)}(x).
\end{aligned}$$

This formula can be verified by multiplying the right hand side of this expansion by $S_{\rho^L}^L(G_2)(x)$ and comparing with the function $S_{\rho^L+\lambda}^L(G_2)(x)$.

5. SHORT ROOT HYBRID CHARACTER FUNCTIONS

Recall from section 2 that, if \mathfrak{g} denotes B_n, C_n, F_4 or G_2 , the short root hybrid character functions can be expressed as a linear combination of the orbit functions

$$\chi_\lambda^S(\mathfrak{g})(x) = \sum_{\mu \in P^+(\mathfrak{g})} q_\mu^\lambda(\mathfrak{g}) C_\mu(\mathfrak{g}).$$

Our goal in this section is to determine the coefficients $q_\mu^\lambda(\mathfrak{g})$ in terms of the readily calculated multiplicities m_μ^λ . We follow the same approach that was used for the long root hybrid character functions.

Let \mathcal{G}_L denote the subgroup of the Weyl group $W(\mathfrak{g})$ generated by the reflections r_α where α ranges over the long simple roots of \mathfrak{g} . We now list a number of properties of this subgroup.

1: By the definition of the Weyl group $W(\mathfrak{g})$ we have that \mathcal{G}_L is a transversal of the Weyl group $W(\Phi_S)$ of the short root subsystem Φ_S in the Weyl group $W(\mathfrak{g})$ —i.e. $W(\mathfrak{g}) = \bigcup_{\phi \in \mathcal{G}_L} W(\Phi_S) \cdot \phi$.

2: By the definition of the sign homomorphism σ^S we have that $\sigma^S(\phi) = +1$ for all $\phi \in \mathcal{G}_L$.

3: There exists a base of simple roots Δ_S in Φ_S such that each $\phi \in \mathcal{G}_L$ permutes these simple roots. In fact we have

\mathfrak{g}	Φ_S	\mathcal{G}_L	Δ_S
B_n	nA_1	$\langle r_{\alpha_1}, \dots, r_{\alpha_{n-1}} \rangle$	$\{e_1, \dots, e_n\}$
C_n	D_n	$\langle r_{\alpha_n} \rangle$	$\{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$
F_4	D_4	$\langle r_\alpha, r_\beta \rangle$	$\{e_2, \frac{1}{2}(e_1 - e_2 - e_3 - e_4), e_3, e_4\}$
G_2	A_2	$\langle r_\alpha \rangle$	$\{\frac{1}{3}(2e_1 - e_2 - e_3), \frac{1}{3}(-e_1 + 2e_2 - e_3)\}$

For each case let $\{\omega''_i\}$ denote the fundamental weights corresponding to the base of simple roots Δ_S . In the table below we provide the expansions for dominant integral weights λ of \mathfrak{g} in terms of the $\{\omega''_i\}$ basis. In particular we observe that each dominant integral weight of \mathfrak{g} is a dominant integral weight of Φ_S . In fact we have

\mathfrak{g}	λ	Φ_S	λ
B_n	$\sum_{i=1}^n N_i \omega_i$	nA_1	$\sum_{i=1}^n 2(N_i + \cdots + \frac{1}{2}N_n)\omega''_i$
C_n	$\sum_{i=1}^n N_i \omega_i$	D_n	$\sum_{i=1}^{n-1} N_i \omega''_i + (N_{n-1} + N_n)\omega''_n$
F_4	$\sum_{i=1}^4 N_i \omega_i$	D_4	$(2N_1 + 2N_2 + N_3)\omega''_1 + N_4\omega''_2 + (2N_2 + N_3)\omega''_3 + N_3\omega''_4$
G_2	$N_1\omega_1 + N_2\omega_2$	A_2	$(3N_1 + N_2)\omega''_1 + N_2\omega''_2$

TABLE 2. Dominant integral weight λ in ω -basis for the Lie algebras \mathfrak{g} is written in ω''_i -basis for Φ_S .

4: Since each $\phi \in \mathcal{G}_L$ permutes the base of simple roots Δ_S , ϕ also permutes the associated fundamental weights $\{\omega''_i\}$ of Φ_S and hence if λ is a dominant integral weight of \mathfrak{g} then $\phi \cdot \lambda$ is also a dominant integral weight of Φ_S . In particular we have that $\phi \cdot \rho^S = \rho^S$.

Using the properties of \mathcal{G}_L listed above we have that for any dominant weight λ of \mathfrak{g}

$$C_\lambda^S(\mathfrak{g})(x) = \sum_{\mu \in \mathcal{G}_L \cdot \lambda} C_\mu(\Phi_S)(x)$$

$$S_{\rho^S + \lambda}^S(\mathfrak{g})(x) = \sum_{\mu \in \mathcal{G}_L \cdot \lambda} S_{\rho^S + \mu}(\Phi_S)(x)$$

and finally

$$\chi_\lambda^S(\mathfrak{g})(x) = \sum_{\mu \in \mathcal{G}_L \cdot \lambda} \chi_\mu(\Phi_S)(x)$$

Since $\chi_\mu(\Phi_S)(x) = \sum_{\nu} m_\nu^\mu(\Phi_S) C_\nu(\Phi_S)(x)$ where $m_\nu^\mu(\Phi_S)$ denotes the multiplicity of the dominant integral weight ν in the simple module of the Lie algebra associated with Φ_S having highest weight μ . Using this we conclude that

$$q_\nu^\lambda(\mathfrak{g}) = \sum_{\mu \in \mathcal{G}_L \cdot \lambda} m_\nu^\mu(\Phi_S). \quad (14)$$

Example 2.

Consider the hybrid character function $\chi_\lambda^S(G_2)(x)$ where $\lambda = 2\omega_1 + \omega_2$. If ω''_i denote the fundamental weights with respect to Δ_S we have that $N_1\omega + N_2\omega_2 = (3N_1 + N_2)\omega''_1 + N_2\omega''_2$ and $r_\alpha \cdot (N_1\omega_1 + N_2\omega_2) = N_2\omega''_1 + (N_3 + N_2)\omega''_2$. For notational convenience we represent the dominant integral weights using coordinates with respect to the fundamental weights of the appropriate algebras. Applying the formula given in (14) we obtain

$$\begin{aligned}
q_{(2,1)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(7,1)}^\nu(A_2) = m_{(7,1)}^{(7,1)}(A_2) + m_{(7,1)}^{(1,7)}(A_2) = 1 + 0 = 1 \\
q_{(1,2)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(5,2)}^\nu(A_2) = m_{(5,2)}^{(7,1)}(A_2) + m_{(5,2)}^{(1,7)}(A_2) = 0 + 1 = 1 \\
q_{(2,0)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(6,0)}^\nu(A_2) = m_{(6,0)}^{(7,1)}(A_2) + m_{(6,0)}^{(1,7)}(A_2) = 0 + 2 = 2 \\
q_{(0,3)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(3,3)}^\nu(A_2) = m_{(3,3)}^{(7,1)}(A_2) + m_{(3,3)}^{(1,7)}(A_2) = 1 + 1 = 2 \\
q_{(1,1)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(4,1)}^\nu(A_2) = m_{(4,1)}^{(7,1)}(A_2) + m_{(4,1)}^{(1,7)}(A_2) = 1 + 2 = 3 \\
q_{(0,2)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(2,2)}^\nu(A_2) = m_{(2,2)}^{(7,1)}(A_2) + m_{(2,2)}^{(1,7)}(A_2) = 2 + 2 = 4 \\
q_{(1,0)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(3,0)}^\nu(A_2) = m_{(3,0)}^{(7,1)}(A_2) + m_{(3,0)}^{(1,7)}(A_2) = 2 + 2 = 4 \\
q_{(0,1)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(1,1)}^\nu(A_2) = m_{(1,1)}^{(7,1)}(A_2) + m_{(1,1)}^{(1,7)}(A_2) = 2 + 2 = 4 \\
q_{(0,0)}^{(2,1)}(G_2) &= \sum_{\nu \in \{(7,1), (1,7)\}} m_{(0,0)}^\nu(A_2) = m_{(0,0)}^{(7,1)}(A_2) + m_{(0,0)}^{(1,7)}(A_2) = 2 + 2 = 4
\end{aligned}$$

These calculation then give

$$\begin{aligned}
\chi_{(2,1)}^S(x) &= C_{(2,1)}(x) + C_{(1,2)}(x) + 2C_{(2,0)}(x) + 2C_{(0,3)}(x) + 3C_{(1,1)}(x) \\
&\quad + 4C_{(0,2)}(x) + 4C_{(1,0)}(x) + 4C_{(0,1)}(x) + 4C_{(0,0)}(x)
\end{aligned}$$

This decomposition can be verified by multiplying the right hand side of the expansion by $S_{\rho^S}^S(G_2)(x)$ and observing that this product yields $S_{\rho^S+\lambda}^S(G_2)(x)$.

6. DOMINANT WEIGHT MULTIPLICITIES FOR THE HYBRID CHARACTERS

In the table below we summarize the dominant weight multiplicities for each of the hybrid characters.

\mathfrak{g}	$p_\mu^\lambda(\mathfrak{g})$	$q_\mu^\lambda(\mathfrak{g})$
B_n	$\sum_{\nu \in \{\lambda, r_{\alpha_n} \cdot \lambda\}} m_\mu^\nu(D_n)$	$\sum_{\nu \in \tilde{S}_n \cdot \lambda} m_\mu^\nu(nA_1)$
C_n	$\sum_{\nu \in S_n \cdot \lambda} m_\mu^\nu(nA_1)$	$\sum_{\nu \in \{\lambda, r_{\alpha_n} \cdot \lambda\}} m_\mu^\nu(D_n)$
F_4	$\sum_{\nu \in \mathcal{G}_S \cdot \lambda} m_\mu^\nu(D_4)$	$\sum_{\nu \in \mathcal{G}_L \cdot \lambda} m_\mu^\nu(D_4)$
G_2	$\sum_{\nu \in \{\lambda, r_\beta \cdot \lambda\}} m_\mu^\nu(A_2)$	$\sum_{\nu \in \{\lambda, r_\alpha \cdot \lambda\}} m_\mu^\nu(A_2)$

TABLE 3. Dominant weight multiplicities $p_\mu^\lambda(\mathfrak{g})$ and $q_\mu^\lambda(\mathfrak{g})$ for hybrid characters $\chi^L(\mathfrak{g})$ and $\chi^S(\mathfrak{g})$ of simple Lie algebra \mathfrak{g} . They are expressed in terms of multiplicities of dominant weights for certain reducible representations of the Lie algebras of long and short roots subsystems of \mathfrak{g} . In the line for the algebra F_4 , \mathcal{G}_L denotes the subalgebra of $W(F_4)$ generated by $\langle r_\alpha, r_\beta \rangle$ and \mathcal{G}_S denotes the subalgebra of $W(F_4)$ generated by $\langle r_\gamma, r_\delta \rangle$.

7. INTERPRETATIONS AND WEYL FORMULAS FOR $\chi_\lambda^L(\mathfrak{g})(0)$ AND $\chi_\lambda^S(\mathfrak{g})(0)$

In the case of the irreducible character function $\chi_\lambda(\mathfrak{g})(x)$ it is well known that $\chi_\lambda(\mathfrak{g})(0)$ is equal to the dimension of the simple \mathfrak{g} module having highest weight λ . In addition we also have the Weyl formula

$$\chi_\lambda(\mathfrak{g})(0) = \frac{\prod(\lambda + \rho, \alpha)}{\prod(\rho, \alpha)}$$

where the products range over the positive roots of the \mathfrak{g} root system.

In this section we provide “dimensional” interpretations and Weyl type formulas for $\chi_\lambda^L(\mathfrak{g})(0)$ and $\chi_\lambda^S(\mathfrak{g})(0)$. We will start with the long root hybrid character functions.

Recall that for the long root hybrid character functions we have

$$\chi_\lambda^L(\mathfrak{g})(x) = \sum_{\mu \in \mathcal{G}_S \cdot \lambda} \chi_\mu(\Phi_L)(x)$$

therefore

$$\chi_\lambda^L(\mathfrak{g})(0) = \sum_{\mu \in \mathcal{G}_S \cdot \lambda} \chi_\mu(\Phi_L)(0)$$

Since the dimension of the simple Φ_L module with highest weight λ , denoted $L_\lambda(\Phi_L)$, is equal to the dimension of the simple Φ_L module with highest weight $\phi \cdot \lambda$ we conclude that

$$\chi_\lambda^L(\mathfrak{g})(0) = |\mathcal{G}_S \cdot \lambda| \dim L_\lambda(\Phi_L) \quad (15)$$

It follows immediately then that

$$\chi_\lambda^L(B_n)(0) = |\mathcal{G}_S \cdot \lambda| \frac{\prod(\lambda + \rho^L(\mathfrak{g}), \alpha)}{\prod(\rho^L(\mathfrak{g}), \alpha)} \quad (16)$$

where the products range over the positive long roots α of \mathfrak{g} .

We have analogous results for the short root characters functions. In fact, recall that for the short root hybrid character functions we have

$$\chi_\lambda^S(\mathfrak{g})(x) = \sum_{\mu \in \mathcal{G}_L \cdot \lambda} \chi_\mu(\Phi_S)(x)$$

therefore

$$\chi_\lambda^S(\mathfrak{g})(0) = \sum_{\mu \in \mathcal{G}_L \cdot \lambda} \chi_\mu(\Phi_S)(0)$$

Since the dimension of the simple Φ_S module with highest weight λ , denoted $L_\lambda(\Phi_S)$, is equal to the dimension of the simple Φ_S module with highest weight $\phi \cdot \lambda$ for any $\phi \in \mathcal{G}_L$, we conclude that

$$\chi_\lambda^S(\mathfrak{g})(0) = |\mathcal{G}_L \cdot \lambda| \dim L_\lambda(\Phi_S) \quad (17)$$

It follows immediately then that

$$\chi_\lambda^S(\mathfrak{g})(0) = |\mathcal{G}_L \cdot \lambda| \frac{\prod(\lambda + \rho^S(\mathfrak{g}), \alpha)}{\prod(\rho^S(\mathfrak{g}), \alpha)} \quad (18)$$

where the products range over the positive short roots α of \mathfrak{g} .

8. CONCLUDING REMARKS

With the determination of the dominant weight multiplicities for the hybrid characters, all but the most curious property of these functions have been revealed. It remains to be seen whether these functions can be interpreted as characters of some algebraic structures.

Combining pairs of the functions C , S forms multivariable generalizations of the common exponential functions [9]. Combining pairs of C , S , S^L , and S^S functions, the number of families of E -functions is increased from one to six families [7].

Discretization of the families of these functions hinges on discretization of the C -functions [6, 5], which is now more than 20 years old [13].

Another class of hybrid characters arises for simple Lie algebras with non-trivial automorphism of their Coxeter-Dynkin diagram. They will be defined and studied elsewhere [10].

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